Published by Institute of Physics Publishing for SISSA/ISAS
Received: *June 12, 2002*
Accepted: *July 29, 2002*

Effective potential on fuzzy sphere

Wung-Hong Huang

Department of Physics, National Cheng Kung University
Tainan, 70101, Taiwan
E-mail: whhwung@mail.ncku.edu.tw

Abstract: The effective potential of quantized scalar field on fuzzy sphere is evaluated to the two-loop level. We see that one-loop potential behaves like that in the commutative sphere and the Coleman-Weinberg mechanism of the radiatively symmetry breaking could be also shown in the fuzzy sphere system. In the two-loop level, we use the heavy-mass approximation and the high-temperature approximation to perform the evaluations. The results show that both of the planar and non-planar Feynman diagrams have inclinations to restore the symmetry breaking in the tree level. However, the contributions from planar diagrams will dominate over those from non-planar diagrams by a factor N^2. Thus, at heavy-mass limit or high-temperature system the quantum field on the fuzzy sphere will behave like those on the commutative sphere. We also see that there is a drastic reduction of the degrees of freedom in the non-planar diagrams when the particle wavelength is smaller than the non-commutativity scale.

Keywords: M-Theory, M(atrix) Theories, Non-Commutative Geometry.

Contents

1. Introduction

Non-commutative fuzzy sphere is known to correspond sphere D2-branes in string theory with background linear B-field [1]. In the presence of constant RR three-form potential the D0-branes are found to expand into a non-commutative fuzzy sphere configurations [2]. It also knows that the field theories on fuzzy sphere appear naturally from D-brane theory and matrix model with some backgrounds [1] and [3]–[6]

In the ordinary matrix model one could not find the fuzzy sphere solutions. However, adding a Chern-Simons term to the matrix model will enable us to describe the non-commutative fuzzy sphere as a classical solution. Comparing the energy in the various classical solutions one can find that the separated D0-branes will expand into a largest non-commutative fuzzy sphere to achieve minimum energy [2].

The problems of the one loop renormalization of the scalar field on the fuzzy sphere have been studied in some recent papers [7]–[10]. The phenomena of the UL/IR mixing [11, 12] on the fuzzy sphere are different from those in the non-commutative plane. Especially, it has found by Chu et al. [9] that , in the limit of the commutative sphere, the two point function is regular without UV/IR mixing; however quantization does not commute with the commutative limit, and a finite “non-commutative anomaly” survives in the commutative limit. Chu et al. [9] use this to provides an explanation of the UV/IR mixing as an infinite variant of the “non-commutative anomaly”.

In this paper we will investigate the effective potential of quantized scalar field [13] on the fuzzy sphere to the two-loop level. In the section 2, we first review the appearance of the fuzzy sphere in the matrix model with a Chern-Simons term. (Note that the Chern-Simons term may be coming from the constant RR three-form potential in the D0-branes system [2].) We then quantize the scalar field on the fuzzy sphere and set up the general formula and Feynman rule to evaluate the effective potential. In the section 3, we see that there is only a single (planar) Feynman diagram in the one-loop potential. The potential behaves like that in the commutative system.

As there appears the Winger $3j$-symbol [14] in Feynman rule we shall adopt some approximations to perform the calculations. In the section 4, we consider the case in which the mass of scalar field is sufficiently large. In the section 5, we consider the case in the high temperature environment [15], which would be that in the early universe. In both cases the two-loop diagrams are evaluated analytically. We see that both of the planar Feynman diagrams and non-planar Feynman diagrams have an inclination to restore the symmetry breaking in the tree level. Also, the contributions from planar diagrams will dominate over those from non-planar diagrams by a factor N^2. We also find that the non-planar diagram will behave as that in the zero space system. This means that there is a drastic reduction of the degrees of freedom in the non-planar diagrams when the wavelength of the scalar particle is smaller than the non-commutativity scale [16, 17, 18]. As a consequence, the quantum fields on a high-temperature non-commutative fuzzy sphere or a heavy quantized field on a non-commutative fuzzy sphere will behave as that on a commutative sphere. The last section is devoted to a short conclusion.

Note that, historically, Fischler et al. [16, 17] had first showed that, at high temperature for which the thermal wavelength is smaller than the non-commutativity scale, there is no way to distinguish and count the contributions of modes to the free energy. Thus, there is a drastic reduction of the degrees of freedom in the non-planar contribution to the thermodynamical potential at high temperature. In [18] we have also confirmed this property by evaluating the two-loop effective potential on a non-commutative plane. We in there investigate the same problem while on a non-commutative fuzzy sphere. This paper is one of our study concerning the quantum field on the non-commutative geometry [18, 19, 20].

2. Quantized scalar field and effective potential on fuzzy sphere

2.1 Fuzzy sphere

The matrix model with Chern-Simons term [1] is described by the action:

$$S = T_0 \,\mathrm{Tr}\left(\frac{1}{2}\dot{X}_i^{\,2} + \frac{1}{4}[X_i, X_j][X_i, X_j] - \frac{i}{3}\lambda_N \epsilon_{ijk} X_i [X_j, X_k]\right), \tag{2.1}$$

where $X_i, i = 1, 2, 3$ are $N \times N$ matrices and $T_0 = \sqrt{2\pi}/g_s$ is the zero-brane tension. The static equations of motion are

$$[X_j, ([X_i, X_j] - i\lambda_N \; \epsilon_{ijk} X_k)] = 0\,, \tag{2.2}$$

and the associated energy is

$$E = -T_0 \,\mathrm{Tr}\left(\frac{1}{4}\,[X_i, X_j]\,[X_i, X_j] - \frac{i}{3}\lambda_N \epsilon_{ijk} X_i\,[X_j, X_k]\right). \tag{2.3}$$

This model admits commutating solutions and static fuzzy sphere solutions [1, 2]. The commutating solutions represent N D0-branes and satisfy the relations

$$[X_i, X_j] = 0\,, \tag{2.4}$$

which have the energy

$$E = 0\,. \tag{2.5}$$

The static fuzzy sphere solutions satisfy the relations

$$[X_i, X_j] = i\lambda_N \epsilon_{ijk} X_k\,, \tag{2.6}$$

and are described by the relations

$$X_i = \lambda_N J_i\,, \tag{2.7}$$

$$X_1^2 + X_2^2 + X_3^2 = R^2\,, \tag{2.8}$$

where J_1, J_2, J_3 define, say, the N-dimensional irreducible representation of SU(2) and are labeled by the spin $\alpha = N/2$. The non-commutativity parameter λ_N is of dimension length, and can be taken positive. The radius R define in (2.8) is quantized in units of λ_N by

$$\frac{R}{\lambda_N} = \sqrt{\frac{N}{2}\left(\frac{N}{2}+1\right)}; \qquad N = 1, 2, \dots\,. \tag{2.9}$$

Besides the above solution X_i may be a direct sum of several irreducible representation of SU(2). Such a configuration could also solve the equation of motion, i.e.

$$X_i = \lambda_N \oplus_{r=1}^{s} J_i^{(r)}, \qquad \sum_{r=1}^{s}(2j_r + 1) = N\,. \tag{2.10}$$

The energy E of these static fuzzy sphere solutions are given by

$$E = -T_0 \lambda^4 \frac{1}{6} \sum_{r=1}^{s} J_r(J_r+1)(2J_r+1)\,. \tag{2.11}$$

From the above relation it is clear that the ground state is the N-dimensional fuzzy sphere [2].

2.2 Quantized scalar field on fuzzy sphere: Feynman rule

A field on the fuzzy sphere is defined as an algebra S_N^2 generated by hermitean operators $\vec{X} = (X_1, X_2, X_3)$ which are described in the section 2.1. The integral of a function $F \in S_N^2$ over the fuzzy sphere is given by [7]–[10]

$$R^2 \int F = \frac{4\pi R^2}{N+1}\,\mathrm{Tr}[F(X)]\,, \tag{2.12}$$

and the inner product can be defined by

$$(F_1, F_2) = \int F_1^\dagger F_2\,. \tag{2.13}$$

A scalar Φ^4 theory on the fuzzy sphere is described by the action

$$S = \int \frac{1}{2}\Phi(\Delta + m^2)\Phi + \frac{g}{4!}\Phi^4 \,. \tag{2.14}$$

Here Φ is a hermitean field, m^2 is the dimensionless mass square, g is a dimensionless coupling and $\Delta = \sum J_i^2$ is the Laplace operator. To evaluate the effective potential of the above model we can use the path-integration formulation of Jackiw [13].

First, we assume that there exists a stationary point at which Φ is a constant field Φ_0. Thus

$$\left.\frac{\delta S}{\delta \Phi}\right|_{\phi_0} = 0 \,. \tag{2.15}$$

Next, we expand the lagrangian about the stationary point and the action becomes

$$S[\Phi] = S[\Phi_0] + \frac{1}{2}\int \tilde{\Phi}(X)\tilde{\Phi}(Y)\left.\frac{\delta^2 S}{\delta\Phi(X)\delta\Phi(Y)}\right|_{\Phi_0} + \int \tilde{\mathcal{L}_{\mathcal{I}}}\left(\tilde{\Phi}, \Phi_0\right), \tag{2.16}$$

in which $\tilde{\Phi} \equiv \Phi - \Phi_0$ and $\tilde{\mathcal{L}_{\mathcal{I}}}(\tilde{\Phi}, \Phi_0)$ can be found from the lagrangian eq. (2.14).

Then, after expanding $\tilde{\Phi}$ in terms of the modes,

$$\tilde{\Phi} = \sum_{L,l} a_l^L Y_l^L \,, \qquad L = 0, 1, \ldots, N \,; \qquad -L \le l \le L \,, \tag{2.17}$$

in which the Fourier coefficient a_l^L are treated as the dynamical variables, the path integral quantization procedure [7] is defined by integrating over all possible configuration of a_l^L. Therefore the k-points Green's functions are computed by the relation

$$\left\langle a_{l_1}^{L_1} \cdots a_{l_k}^{L_k} \right\rangle = \frac{\int [D\tilde{\Phi}] e^{-S} a_{l_1}^{L_1} \cdots a_{l_k}^{L_k}}{\int [D\tilde{\Phi}] e^{-S}} \,. \tag{2.18}$$

Note that the complete basis of functions on S_N^2 is given by the $(N{+}1)^2$ spherical harmonics, $Y_l^L, (L = 0, 1, \ldots, N; -L \le l \le L)$. They correspond to the usual spherical harmonics, however the angular momentum has an upper bound N here. This is a characteristic feature of fuzzy sphere.

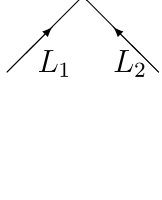
The propagator so obtained is

$$D^{-1}(\Phi_0) = \left\langle a_l^L a_{l'}^{L'\dagger} \right\rangle = (-1)^l \left\langle a_l^L a_{-l'}^{L'} \right\rangle = \delta_{LL'}\delta_{ll'} \frac{1}{L(L+1) + \mu^2} \,, \tag{2.19}$$

in which $a_l^{L\dagger} = (-1)^l a_{-l}^L$ and

$$\mu^2 = m^2 + \frac{1}{2} g \Phi_0^2 \,. \tag{2.20}$$

The four-legs vertices are given by

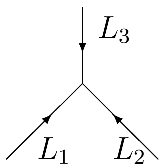


$$a_{l_1}^{L_1} \cdots a_{l_4}^{L_4}\; V_4(L_1, l_1; \cdots; L_4, l_4) \tag{2.21}$$

where

$$V_4(L_1,l_1;\ldots;L_4,l_4) = \frac{g}{4!}\frac{N+1}{4\pi}(-1)^{L_1+L_2+L_3+L_4}\prod_{i=1}^{4}(2L_i+1)^{1/2}\times$$
$$\times\sum_{L,l}(-1)^l(2L+1)\begin{pmatrix} L_1 & L_2 & L \\ l_1 & l_2 & l \end{pmatrix}\times$$
$$\times\begin{pmatrix} L_3 & L_4 & L \\ l_3 & l_4 & -l \end{pmatrix}\begin{Bmatrix} L_1 & L_2 & L \\ \alpha & \alpha & \alpha \end{Bmatrix}\begin{Bmatrix} L_3 & L_4 & L \\ \alpha & \alpha & \alpha \end{Bmatrix}. \tag{2.22}$$

The above Feynman rule of propagator and four-legs vertices can be found in [9]. Here we need also the Feynman rule of three-legs vertices, it is described as



$$a^{L_1}_{l_1}a^{L_2}_{l_2}a^{L_3}_{l_3}\;V_3(L_1,l_1;L_2,l_2;L_3,l_3) \tag{2.23}$$

where

$$V_3(L_1,l_1;L_2,l_2;L_3,l_3) = \frac{g}{6}\Phi_0\sqrt{\frac{N+1}{4\pi}}(-1)^{2\alpha+L_1+L_2+L_3}\sqrt{(2L_1+1)(2L_2+1)(2L_3+1)}\times$$
$$\times\begin{pmatrix} L_1 & L_2 & L_3 \\ l_1 & l_2 & -l_3 \end{pmatrix}\begin{Bmatrix} L_1 & L_2 & L_3 \\ \alpha & \alpha & \alpha \end{Bmatrix}. \tag{2.24}$$

Here the first bracket is the Wigner $3j$-symbol and the curly bracket is the $6j$-symbol of SU(2), in the standard mathematical normalization [14]. Note that to derive the above Feynman rule we have used the following "fusion" algebra [14]

$$Y^I_iY^J_j = \sqrt{\frac{N+1}{4\pi}}\sum_{K,k}(-1)^{2\alpha+I+J+K+k}\sqrt{(2I+1)(2J+1)(2K+1)}\times$$
$$\times\begin{pmatrix} I & J & K \\ i & j & -k \end{pmatrix}\begin{Bmatrix} I & J & K \\ \alpha & \alpha & \alpha \end{Bmatrix}Y^K_k\,, \tag{2.25}$$

where the sum is over $0\le K\le N, -K\le k\le K$, and $\alpha = N/2$.

Then the effective potential $V(\phi_0)$ is found to be [13]

$$V(\Phi_0) = V_0(\Phi_0)+\frac{1}{2}\int \ln\det\left[D^{-1}(\Phi_0)\right]-\left\langle \exp\left(\int\tilde{\mathcal{L}_{\mathcal{I}}}(\tilde{\Phi},\Phi_0)\right)\right\rangle. \tag{2.26}$$

The first term in eq. (2.20) is the classical potential. The second term is the one-loop contribution from the second term in eq. (2.16). The third term is the higher-loop contribution of the effective potential. To obtain it we shall evaluate the expectation value of the third term in eq. (2.16) by the conventional Feynman rule, with $D^{-1}(\Phi_0)$ as the propagator and keep only the connected single-particle irreducible graphs [13].

In next section, we use the above formula to analyze the one-loop potential, and in the sections 4 and 5, we detailed evaluate the two-loop potential on the fuzzy sphere.

JHEP07(2002)064

3. One-loop effective potential

Now, using the formula in (2.26) the one-loop effective potential on a fuzzy sphere becomes

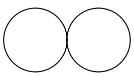
$$\begin{aligned}\frac{1}{2}\int \ln\det\left[D^{-1}(\Phi_0,\vec{k})\right] &= -\frac{1}{2}\frac{1}{N+1}\sum_{L,l}\ln\left[L(L+1)+\mu^2\right]\\ &= -\frac{1}{2}\frac{1}{N+1}\sum_{L=0}^{N}(2L+1)\ln\left[L(L+1)+\mu^2\right]\\ &= -\frac{1}{2}\frac{1}{N+1}\ln\left(m^2+\frac{1}{2}g\Phi_0^2\right)-\\ &\quad -\frac{1}{2}\frac{1}{N+1}\sum_{L=1}^{N}\ln\left[L(L+1)+m^2+\frac{1}{2}g\Phi_0^2\right].\end{aligned}\tag{3.1}$$

From the above result we can see the following properties. In the massless case, we expand the second term around the point $\Phi_0^2=0$ and it is easily to see that, they will renormalize the mass and coupling constant and give the high-power terms of Φ_0^2. However, the first term will become negative for sufficiently large Φ_0. Thus it could break the symmetry and the Coleman-Weinberg mechanism is shown. In the massive case, we can expand eq. (3.1) around the point $\Phi_0^2=0$ and it is easily to see that, the one-loop result could only renormalize the mass and coupling constant, and give the high-power terms of Φ_0^2, but they could not change the symmetry property in the tree level.

The above property is independent of N and thus will behave like that in the commutative sphere system. Note that as $N\to\infty$ the fuzzy sphere will become a commutative sphere.

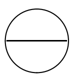
4. Two-loop effective potential: heave mass approximation

In the two-loop level we have to evaluate the following two planar and two non-planar diagrams. The contributions of the effective potential from the planar diagram are



$$= I_1^P$$

$$= -\frac{g}{4\pi}\frac{1}{3}\frac{1}{(N+1)^2}\sum_L\frac{2L+1}{L(L+1)+\mu^2}\sum_J\frac{2J+1}{J(J+1)+\mu^2}\,,\tag{4.1}$$



$$= I_2^P$$

$$\begin{aligned}&= -\Phi_0^2\left(\frac{g}{4\pi}\right)^2\frac{1}{6}\frac{1}{(N+1)^2}\times\\ &\times\sum_{L,J}\frac{2L+1}{L(L+1)+\mu^2}\frac{2J+1}{J(J+1)+\mu^2}\frac{2L+2J+1}{(L+J)(L+J+1)+\mu^2}.\end{aligned}\tag{4.2}$$

The contributions of th effective potential from the non-planar diagram are like those in the planar diagram, while with an extra factor. They are

$$\begin{aligned} I_1^N &= -\frac{g}{4\pi}\frac{1}{6}\frac{1}{(N+1)^2}\sum_{L,J=0}^{N}\frac{2L+1}{L(L+1)+\mu^2}\frac{2J+1}{J(J+1)+\mu^2}\times \\ &\qquad\times(-1)^{L+J+2\alpha}(2\alpha+1)\begin{Bmatrix}\alpha & \alpha & L\\ \alpha & \alpha & J\end{Bmatrix}, \end{aligned} \tag{4.3}$$

$$\begin{aligned} I_2^N &= -\Phi_0^2\left(\frac{g}{4\pi}\right)^2\frac{1}{6}\frac{1}{(N+1)^2}\times \\ &\quad\times\sum_{L,J}\frac{2L+1}{L(L+1)+\mu^2}\frac{2J+1}{J(J+1)+\mu^2}\times \\ &\qquad\times\frac{2L+2J+1}{(L+J)(L+J+1)+\mu^2}(-1)^{L+J+2\alpha}(2\alpha+1)\begin{Bmatrix}\alpha & \alpha & L\\ \alpha & \alpha & J\end{Bmatrix}. \end{aligned} \tag{4.4}$$

To derive the above relations we have use the following identities of the $3j$ and $6j$ symbols, which can be found in [14]: The $3j$ symbols satisfy the orthogonality relation

$$\sum_{j,l}\begin{pmatrix}J & L & K\\ j & l & k\end{pmatrix}\begin{pmatrix}J & L & K'\\ -j & -l & -k'\end{pmatrix}=\frac{(-1)^{K-L-J}}{2K+1}\delta_{K,K'}\delta_{k,k'}, \tag{4.5}$$

assuming that (J,L,K) form a triangle. The $6j$ symbols satisfy the orthogonality relation

$$\sum_N(2N+1)\begin{Bmatrix}A & B & N\\ C & D & P\end{Bmatrix}\begin{Bmatrix}A & B & N\\ C & D & Q\end{Bmatrix}=\frac{1}{2P+1}\delta_{P,Q}, \tag{4.6}$$

and the following sum rule

$$\sum_N(-1)^{N+P+Q}(2N+1)\begin{Bmatrix}A & B & N\\ C & D & P\end{Bmatrix}\begin{Bmatrix}A & B & N\\ D & C & Q\end{Bmatrix}=\begin{Bmatrix}A & C & Q\\ B & D & P\end{Bmatrix}, \tag{4.7}$$

assuming that (A,D,P) and (B,C,P) form a triangle.

To proceed the calculation we shall adopt some approximations because it is difficult to perform summation over the index in the $6j$ symbols.

Note that to evaluate the scalar field propagator, Chu et al. [9] have considered the case that the external moment $L \ll \alpha$. In this case they used the relation [14]

$$\begin{Bmatrix}\alpha & \alpha & L\\ \alpha & \alpha & J\end{Bmatrix}\approx(-1)^{\frac{L+J+2\alpha}{2\alpha}}P_L\left(1-\frac{J^2}{2\alpha^2}\right),\qquad L\ll\alpha, \tag{4.8}$$

to evaluate the one-loop correction to the mass square. They have found that the non-commutative anomaly, which will survive in the commutative sphere limit, is $-\frac{g}{12\pi}\sum_{k=1}^{L}\frac{1}{k}$.

As we will perform the two-loop calculation in which the summations in (4.3) and (4.4) are over all the values of L, $0\le L\le N$, the approximation of (4.8) can not be used now. To proceed, we will perform the evaluations of the two-loop diagrams under the approximation that $m^2 \gg N$.

It is easy to perform the approximation in the planar diagrams. The results are

$$\begin{aligned}
I_1^P &= -\frac{g}{4\pi}\frac{1}{3}\frac{1}{(N+1)^2}\sum_{L,J=0}^{N}\frac{2L+1}{\mu^2}\left(1-\frac{L(L+1)}{\mu^2}+\cdots\right)\frac{2J+1}{\mu^2}\left(1-\frac{J(J+1)}{\mu^2}+\cdots\right) \\
&= -\frac{g}{4\pi}\frac{1}{3}\frac{1}{(N+1)^2}\left[\frac{(N+1)^4}{\mu^4}-\frac{N(N+2)(N+1)^4}{\mu^6}+O\left(\frac{N^8}{\mu^8}\right)\right], \qquad (4.9)
\end{aligned}$$

$$I_2^P = -\Phi_0^2\left(\frac{g}{4\pi}\right)^2\frac{1}{6}\frac{1}{(N+1)^2}\left[\frac{1}{\mu^6}\left(\frac{2}{3}N(N+1)^2(N+5)+4N\right)+O\left(\frac{N^8}{\mu^8}\right)\right]. \qquad (4.10)$$

The contributions of non-planar diagrams are calculated as follows.

$$\begin{aligned}
I_1^N &= -\frac{g}{4\pi}\frac{1}{6}\frac{1}{(N+1)^2}\sum_{L,J=0}^{N}\frac{2L+1}{\mu^2}\left(1-\frac{L(L+1)}{\mu^2}+\cdots\right)\frac{2J+1}{\mu^2}\left(1-\frac{J(J+1)}{\mu^2}+\cdots\right)\times \\
&\quad\times(-1)^{L+J+2\alpha}(2\alpha+1)\begin{Bmatrix}\alpha & \alpha & L\\ \alpha & \alpha & J\end{Bmatrix} \\
&= -\frac{g}{4\pi}\frac{1}{6}\frac{1}{(N+1)^2}\left[\frac{(N+1)^2}{\mu^4}+O\left(\frac{N^6}{\mu^6}\right)\right]. \qquad (4.11)
\end{aligned}$$

To obtain the above results we have used the formula [14]

$$\sum_L (2L+1)(-1)^{L+2\alpha}\begin{Bmatrix}\alpha & \alpha & L\\ \alpha & \alpha & J\end{Bmatrix} = \delta_{J,0}(2\alpha+1)\,. \qquad (4.12)$$

Now using the above formula and a simple relation $(2L+1)(2J+1)(2L+2J+1) = 4(2L+1)J(J+1)+(2L+1)+2L(2L+1)(2J+1)$ we have another result

$$\begin{aligned}
I_2^N &= -\Phi_0^2\left(\frac{g}{4\pi}\right)^2\frac{1}{6}\frac{1}{(N+1)^2}\times \\
&\quad\times\sum_{L,J}\frac{2L+1}{\mu^2}\left(1-\frac{L(L+1)}{\mu^2}+\cdots\right)\frac{2J+1}{\mu^2}\left(1-\frac{J(J+1)}{\mu^2}+\cdots\right)\times \\
&\quad\times\frac{2L+2J+1}{\mu^2}\left(1-\frac{(L+J)(L+J+1)}{\mu^2}+\cdots\right)\times \\
&\quad\times(-1)^{L+J+2\alpha}(2\alpha+1)\begin{Bmatrix}\alpha & \alpha & L\\ \alpha & \alpha & J\end{Bmatrix} \\
&= -\Phi_0^2\left(\frac{g}{4\pi}\right)^2\frac{1}{6}\frac{1}{(N+1)^2}\left[\frac{(N+1)^2}{\mu^6}+O\left(\frac{N^8}{\mu^8}\right)\right]. \qquad (4.13)
\end{aligned}$$

From the above results we find that

$$\begin{aligned}
I_1^N+I_2^N+I_1^P+I_2^P &= -\frac{g}{4\pi}\frac{1}{6}\left[\frac{1+2(N+1)^2}{\mu^4}+O\left(\frac{N^6}{\mu^6}\right)\right] \\
&= \frac{g}{4\pi}\frac{1}{6m^4}\left[2(N+1)^2+1\right]\left(-1+g\Phi_0^2+\cdots\right)+O\left(\frac{N^6}{m^6}\right). \qquad (4.14)
\end{aligned}$$

Thus, the two-loop effect potential is to give a positive correction to the mass square term and has an inclination to restore the symmetry breaking in the tree level. (Notice that we set the scale to be $4\pi R^2 = 1$.)

Although the heavy mass approximation adopted in this section enables us to evaluate the two-loop effective potential analytically, the approximation is very constricted. In the next section we will investigate the system in the high temperature environment, which would be that in the early universe and thus is a realistic problem. Under this approximation we could evaluate the two-loop effective potential analytically and, besides, we find an interesting property of the drastic reduction of the degrees of freedom in the non-planar diagram.

5. Two-loop effective potential: high temperature approximation

To consider the finite temperature system the action in (2.14) shall be replaced by

$$S = \int \frac{1}{2}\Phi\left(\partial_t^2 + \Delta + m^2\right)\Phi + \frac{g}{4!}\Phi^4 \,. \tag{5.1}$$

The quantized field $\tilde{\Phi}$ in (2.7) shall be expand in terms of the modes in the following relation

$$\tilde{\Phi} = \int_{-\infty}^{\infty} \frac{dp_0}{(2\pi)} \sum_{L,l} e^{ip_0 t} a_l^L(p_0) Y_l^L \,, \qquad L = 0, 1, \ldots, N \,; \qquad -L \le l \le L \,, \tag{5.2}$$

in which the Fourier coefficient $a_l^L(p_0)$ are treated as the dynamical variables and Y_l^L are the usual spherical harmonics. The propagator so obtained is

$$D^{-1}(\Phi_0) = \left\langle a(p_0)_l^L a(p_0)_{l'}^{L'\dagger} \right\rangle = \delta_{LL'}\delta_{ll'} \frac{1}{p_0^2 + L(L+1) + \mu^2} \,, \tag{5.3}$$

in which we have let the spacetime to be the euclidean type. Then for the theory at finite temperature $T = 1/\beta$ we can take the following substitutions [15]–[18]:

$$p_0 \longrightarrow \frac{2\pi p_0}{\beta} \,, \tag{5.4}$$

$$\int dp_0 \longrightarrow \frac{2\pi}{\beta} \sum_{p_0} . \tag{5.5}$$

in which p_0 is an integral.

Now, we can use the previous prescriptions to evaluate the finite temperature two-loop effective potential. The contributions of planar diagrams are

$$\begin{aligned} I_1^P &= -\frac{g}{12\pi}\frac{1}{(N+1)^2}\left(\frac{2\pi}{\beta}\right)^2 \times \\ &\quad \times \sum_{p_0}\left[\frac{(N+1)^4}{[(2\pi p_0/\beta)^2+\mu^2]^2} - \frac{N(N+2)(N+1)^4}{[(2\pi p_0/\beta)^2+\mu^2]^3} + O\left(\frac{N^8}{[(2\pi p_0/\beta)^2+\mu^2]^4}\right)\right] \\ &= -\frac{g}{12\pi}\frac{(N+1)^2}{\mu^4}\left(\frac{2\pi}{\beta}\right)^2 + O\left((\beta N)^2\right) , \end{aligned} \tag{5.6}$$

$$
\begin{aligned}
I_2^P &= -\Phi_0^2 \left(\frac{g}{4\pi}\right)^2 \frac{1}{6}\frac{1}{(N+1)^2}\left(\frac{2\pi}{\beta}\right)^2 \times \\
&\quad \times \sum_{p_0} \left[\frac{\frac{2}{3}N(N+1)^3(N+5)+4N}{[(2\pi p_0/\beta)^2+\mu^2]^3} + O\left(\frac{N^8}{[(2\pi p_0/\beta)^2+\mu^2]^4}\right)\right] \\
&= -\Phi_0^2 \left(\frac{g}{4\pi}\right)^2 \frac{1}{9}\frac{N(N+1)^2(N+5)+6N}{(N+1)^2\mu^6}\left(\frac{2\pi}{\beta}\right)^2 + O\left((\beta N)^2\right). \qquad (5.7)
\end{aligned}
$$

The contributions of non-planar diagrams are

$$
\begin{aligned}
I_1^N &= -\frac{g}{4\pi}\frac{1}{6}\frac{1}{(N+1)^2}\left(\frac{2\pi}{\beta}\right)^2 \sum_{p_0}\left[\frac{(N+1)^2}{[(2\pi p_0/\beta)^2+\mu^2]^2} + O\left(\frac{N^6}{[(2\pi p_0/\beta)^2+\mu^2]^4}\right)\right] \\
&= -\frac{g}{24\pi}\frac{1}{\mu^4}\left(\frac{2\pi}{\beta}\right)^2 + O\left((\beta N)^2\right), \qquad (5.8) \\
I_2^N &= -\Phi_0^2\left(\frac{g}{4\pi}\right)^2 \frac{1}{6}\frac{1}{(N+1)^2}\left(\frac{2\pi}{\beta}\right)^2 \sum_{p_0}\left[\frac{(N+1)^2}{[(2\pi p_0/\beta)^2+\mu^2]^3} + O\left(\frac{N^6}{[(2\pi p_0/\beta)^2+\mu^2]^4}\right)\right] \\
&= -\Phi_0^2\left(\frac{g}{4\pi}\right)^2 \frac{1}{6}\frac{1}{\mu^6}\left(\frac{2\pi}{\beta}\right)^2 + O\left((\beta N)^2\right). \qquad (5.9)
\end{aligned}
$$

The above calculations are just those in (4.9), (4.10), (4.11) and (4.13) while added the summation factor (5.5) and replaced μ^2 by $(2\pi p_0/\beta)^2 + \mu^2$. The terms in $O((\beta N)^2)$, which may be evaluated by the zeta-function regularization method, become negligible at a sufficiently high temperature such that $\beta \ll N$. The results are similar to those in the section 4 and thus both of the planar and non-planar Feynman diagrams have inclinations to restore the symmetry breaking in the tree level.

Comparing the eqs. (5.6)–(5.9) we see that when $N \gg 1$ the contributions from planar diagrams will dominate over those from non-planar diagrams by a factor N^2. Thus our results indicate that, at high temperature ($\beta N \ll 1$) the quantum field on the fuzzy sphere will behave like those on the commutative sphere. (Note that we set the scale to be $4\pi R^2 = 1$.)

It is also worthy to notice that the leading contribution from non-planar diagrams, i.e. (5.8) and (5.9), does not depend on N. This property can be interpreted is as a drastic reduction of the degrees of freedom in the non-planar diagrams when the thermal wavelength is smaller than the non-commutativity scale. In fact, the evaluations about the high-temperature two-loop potential at zero space would enable us to see that the potentials are similar to (5.8) and (5.9). (The calculation is a easy work as there is only a planar diagram.)

We now explain the property of reduction of degrees of freedom more clear as following [16, 17, 18]. It is known that as $N \to \infty$ the fuzzy sphere will become a commutative sphere. Also, when

$$
N \longrightarrow \infty, R^2 = \frac{N\theta}{2} \longrightarrow \infty\,, \qquad \text{keeping } \theta \text{ fixed}. \qquad (5.10)
$$

we will have a non-commutative plane theory [9]. Then, using the definition of the thermal wavelength $\lambda = \hbar\sqrt{2\pi\beta/m}$, the high temperature approximation is useful under the condition

$$1 > \beta N = \frac{\lambda^2 m}{2\pi\hbar^2} N\,. \tag{5.11}$$

Now, put back the scale $4\pi R^2(=1)$ and reset the scale to be $m/2\pi\hbar^2 = 1$, the above relation becomes $4\pi R^2 > \lambda^2 N$. Finally, using the definition of θ in (5.10) we see that the high temperature condition becomes.

$$2\pi\theta > \lambda^2\,. \tag{5.12}$$

This means that there is a drastic reduction of the degrees of freedom in the non-planar diagrams when the thermal wavelength is smaller than the non-commutativity scale.

6. Conclusion

We have studied the effective potential of quantized scalar field on fuzzy sphere to the two-loop level. We see that one-loop potential will behave like that in the commutative sphere and the Coleman-Weinberg mechanism of the radiatively symmetry breaking could be also shown in the fuzzy sphere system.

In the two-loop potential, both of the planar and non-planar Feynman diagrams can appear. We use the heavy-mass approximation and high-temperature approximation to perform the calculations analytically. We see that both of the planar Feynman diagrams and non-planar Feynman diagrams have an inclination to restore the symmetry breaking in the tree level. We also see that the contributions from planar diagrams will dominate over those from non-planar diagrams by a factor N^2.

We also find that the non-planar diagram will behave as that in the zero space system. This means that there is a drastic reduction of the degrees of freedom in the non-planar diagrams when the thermal wavelength is smaller than the non-commutativity scale. Thus our results indicate that, at high temperature ($\beta N \ll 1$) the quantum field on the fuzzy sphere will behave like those on the commutative sphere.

In this paper we have only considered the non-commutative scalar field theory. The more realistic model including the gauge fields on a fuzzy sphere [5, 6, 21] is deserved studied. However, we believe that the property of the reduction of the degrees of freedom in the non-planar diagram at high temperature will also be shown in other models. It remains to be investigated.

References

[1] A.Y. Alekseev, A. Recknagel and V. Schomerus, *Non-commutative world-volume geometries: branes on* SU(2) *and fuzzy spheres*, *J. High Energy Phys.* **09** (1999) 023 [hep-th/9908040]; *Brane dynamics in background fluxes and non-commutative geometry*, *J. High Energy Phys.* **05** (2000) 010 [hep-th/0003187].

[2] R.C. Myers, *Dielectric-branes*, *J. High Energy Phys.* **12** (1999) 022 [hep-th/9910053].

[3] P.-M. Ho, *Fuzzy sphere from matrix model*, *J. High Energy Phys.* **12** (2000) 015 [hep-th/0010165].

[4] C. Bachas, M.R. Douglas and C. Schweigert, *Flux stabilization of D-branes*, *J. High Energy Phys.* **05** (2000) 048 [hep-th/0003037].

[5] K. Hashimoto and K. Krasnov, *D-brane solutions in non-commutative gauge theory on fuzzy sphere*, *Phys. Rev.* **D 64** (2001) 046007 [hep-th/0101145];
S. Iso, Y. Kimura, K. Tanaka and K. Wakatsuki, *Noncommutative gauge theory on fuzzy sphere from matrix model*, *Nucl. Phys.* **B 604** (2001) 121 [hep-th/0101102].

[6] Y. Kimura, *Noncommutative gauge theories on fuzzy sphere and fuzzy torus from matrix model*, *Prog. Theor. Phys.* **106** (2001) 445 [hep-th/0103192].

[7] H. Grosse, C. Klimčík and P. Prešnajder, *Towards finite quantum field theory in noncommutative geometry*, *Int. J. Theor. Phys.* **35** (1996) 231 [hep-th/9505175].

[8] S. Vaidya, *Perturbative dynamics on fuzzy S^2 and RP^2*, *Phys. Lett.* **B 512** (2001) 403 [hep-th/0102212].

[9] C.-S. Chu, J. Madore and H. Steinacker, *Scaling limits of the fuzzy sphere at one loop*, *J. High Energy Phys.* **08** (2001) 038 [hep-th/0106205].

[10] B.P. Dolan, D. O'Connor and P. Prešnajder, *Matrix Φ^4 models on the fuzzy sphere and their continuum limits*, *J. High Energy Phys.* **03** (2002) 013 [hep-th/0109084].

[11] S. Minwalla, M. Van Raamsdonk and N. Seiberg, *Noncommutative perturbative dynamics*, *J. High Energy Phys.* **02** (2000) 020 [hep-th/9912072].

[12] M.R. Douglas and N.A. Nekrasov, *Noncommutative field theory*, *Rev. Mod. Phys.* **73** (2001) 977 [hep-th/0106048].

[13] R. Jackiw, *Functional evaluation of the effective potential*, *Phys. Rev.* **D 9** (1974) 1686.

[14] D.A. Varshalovich, A.N. Moskalev and V.K. Khersonsky, *Quantum theory of angular momentum: irreducible tensors, spherical harmonics, vector coupling coefficients, 3nj symbols*, World Scientific, Singapore 1988.

[15] L. Dolan and R. Jackiw, *Symmetry behavior at finite temperature*, *Phys. Rev.* **D 9** (1974) 3320.

[16] W. Fischler, E. Gorbatov, A. Kashani-Poor, S. Paban, P. Pouliot and J. Gomis, *Evidence for winding states in noncommutative quantum field theory*, *J. High Energy Phys.* **05** (2000) 024 [hep-th/0002067].

[17] W. Fischler, E. Gorbatov, A. Kashani-Poor, R. McNees, S. Paban, P. Pouliot and J. Gomis, *The interplay between θ and T*, *J. High Energy Phys.* **06** (2000) 032 [hep-th/0003216].

[18] W.H. Huang, *High-temperature effective potential of noncommutative scalar field theory: reduction of degree of freedom by noncommutativity*, *Phys. Rev.* **D 63** (2001) 125004 [hep-th/0101040]; *Two-loop effective potential in noncommutative scalar field theory*, *Phys. Lett.* **B 496** (2000) 206 [hep-th/0009067].

[19] W.H. Huang, *Finite-temperature casimir effect on the radius stabilization of noncommutative torus*, *J. High Energy Phys.* **11** (2000) 041 [hep-th/0011037]; *Casimir effect on the radius stabilization of the noncommutative torus*, *Phys. Lett.* **B 497** (2001) 317 [hep-th/0010160].

[20] W.H. Huang, *Quantum stabilization of compact space by extra fuzzy space*, *Phys. Lett.* **B 537** (2002) 311 [hep-th/0203176].

[21] U. Carow-Watamura and S. Watamura, *Noncommutative geometry and gauge theory on fuzzy sphere*, *Commun. Math. Phys.* **212** (2000) 395 [hep-th/9801195].

JHEP07(2002)064